Norm and trace Inequalities for positive semidefinite matrices

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In this talk, we present several unitarily invariant norm and trace inequalities for positive semidefinite matrices. These inequalities are related to:

**Part 1:** A question of Bourin

**Part 2:** The arithmetic-geometric mean inequality and the Heinz inequality

**Part 3:** The Ando-Hiai-Okubo trace inequality

In the sequel, $|||\cdot|||$ denotes any unitarily invariant norm.
In his work on the subadditivity of concave functions of positive semidefinite matrices, Bourin (2009) asked the following question: Given $A, B \geq 0$ and $p, q > 0$, is it true that
\[
\| A^p B^q + B^p A^q \| \leq \| A^{p+q} + B^{p+q} \| ?
\]

The related well-known Heinz inequality says that
\[
\| A^p B^q + A^q B^p \| \leq \| A^{p+q} + B^{p+q} \| .
\]

Hayajneh and Kittaneh (2013) settled Bourin’s question affirmatively for $p = 1, 2, 3$ and $q = 1$ in the case of the Hilbert–Schmidt norm $\| \cdot \|_2$. 
Replacing $A$ and $B$ by $A^{\frac{1}{p+q}}$ and $B^{\frac{1}{p+q}}$, we see that the inequality in Bourin’s question and the Heinz inequality are equivalent to the inequalities:

$$\|A^t B^{1-t} + B^t A^{1-t}\| \leq \|A + B\|$$

and

$$\|A^t B^{1-t} + A^{1-t} B^t\| \leq \|A + B\|$$

for $0 \leq t \leq 1$, respectively.

Bhatia (2014) proved that

$$\|A^t B^{1-t} + B^t A^{1-t}\|_2 \leq \|A + B\|_2$$

for $t \in \left[\frac{1}{4}, \frac{3}{4}\right]$, which is an extension of the result of Hayajneh and Kittaneh.
In order to solve Bourin’s question positively, Hayajneh and Kittaneh (2013) conjectured that

\[ ||| A^p B^q + B^p A^q ||| \leq ||| A^p B^q + A^q B^p ||| \]

Equivalently,

\[ ||| A^t B^{1-t} + B^t A^{1-t} ||| \leq ||| A^t B^{1-t} + A^{1-t} B^t ||| \]

for \( 0 \leq t \leq 1 \).

In the case of the Hilbert–Schmidt norm, we see that inequality

\[ || A^t B^{1-t} + B^t A^{1-t} ||_2 \leq || A^t B^{1-t} + A^{1-t} B^t ||_2 \]

is equivalent to saying

\[ \text{Re } \text{tr } A^t B^{1-t} A^{1-t} B^t \leq \text{tr } AB. \]
Bhatia (2014) proved the inequality

\[ \Re \operatorname{tr} A^t B^{1-t} A^{1-t} B^t \leq \operatorname{tr} AB \]

for \( t \in \left[ \frac{1}{4}, \frac{3}{4} \right] \).

Bottazzi et al. (2015) gave a counterexample to show that for the usual operator norm \( \| \cdot \| \), the inequality

\[ \| A^t B^{1-t} + B^t A^{1-t} \| \leq \| A^t B^{1-t} + A^{1-t} B^t \| \]

does not hold for all \( t \in [0, 1] \).
In (1993), Kittaneh proved the Heinz-Kato type inequality

$$||| A^t X B^{1-t} ||| \leq ||| A X |||^t ||| X B |||^{1-t}$$

for every matrix $X$ and for $0 \leq t \leq 1$.

The classical Young’s inequality for scalars says that if $a, b$ are positive real numbers, then

$$a^t b^{1-t} \leq ta + (1 - t) b$$

for $0 \leq t \leq 1$. Thus, we have the scalar Heinz inequality

$$a^t b^{1-t} + a^{1-t} b^t \leq a + b$$

for $0 \leq t \leq 1$. 
Based on the triangle inequality, the Heinz-Kato type inequality, the scalar Heinz inequality, and the self-adjointness of unitarily invariant norms, Hayajneh et al. (2017) proved that

\[ \| A^t X B^{1-t} + B^t X^* A^{1-t} \| \leq \| AX \| + \| XB \|. \]

As an application of the previous inequality for the trace norm \( \| \cdot \|_1 \), we have

\[ \| A^t B^{1-t} + B^t A^{1-t} \|_1 \leq \| A + B \|_1. \]

This gives an affirmative answer to Bourin’s question in the case of the trace norm.

However, the question of Bourin remains open for other unitarily invariant norms like the Schatten \( p \)-norms.
Hayajneh et al. (2017) proved that

\[ \| \text{Re} \left( A^t B^{1-t} + B^t A^{1-t} \right) \| \leq \| A + B \| \]

and

\[ \| \text{Im} \left( A^t B^{1-t} + B^t A^{1-t} \right) \| \leq \| A + B \| . \]

These inequalities give a partial answer to Bourin’s question by inserting the real and imaginary parts in the left-hand side of the inequality

\[ \| A^t B^{1-t} + B^t A^{1-t} \| \leq \| A + B \| . \]
Part 2: The arithmetic-geometric mean inequality and the Heinz inequality

The arithmetic-geometric mean inequality for scalars can be refined by inserting the Heinz mean between the geometric mean and the arithmetic mean:

\[ \sqrt{ab} \leq \frac{a^t b^{1-t} + a^{1-t} b^t}{2} \leq \frac{a + b}{2}, \]

where \( a, b \) are positive real numbers and \( 0 \leq t \leq 1 \). These inequalities are equivalent to the inequalities

\[ \left( a^{1/2} + b^{1/2} \right)^2 \leq (a^t + b^t) \left( a^{1-t} + b^{1-t} \right) \leq 2 (a + b). \]
Let
\[ h_t = A^t B^{1-t} + A^{1-t} B^t, \]
\[ b_t = A^t B^{1-t} + B^t A^{1-t}, \]
\[ k_t = (A^t + B^t) (A^{1-t} + B^{1-t}), \]
and
\[ m_t = (A^{1-t} + B^{1-t})^{1/2} (A^t + B^t) (A^{1-t} + B^{1-t})^{1/2}. \]

Then the Heinz matrix versions of the previous scalar inequalities are:
\[ 2 \left\| A^{1/2} B^{1/2} \right\| \leq \left\| A^t B^{1-t} + A^{1-t} B^t \right\| \leq \left\| A + B \right\|, \]
which can be stated as
\[ 2 \left\| h_{1/2} \right\| \leq \left\| h_t \right\| \leq \left\| h_1 \right\|. \]
For the usual operator norm, this was proved by Heinz (1951) and the generalization to all unitarily invariant norms was obtained by Bhatia and Davis (1993).

Because of the noncommutativity of matrix multiplication, $h_t$ is not the same as $b_t$, and this gives us the ability to see the arithmetic-geometric mean inequality in a different new way in terms of $b_t$, which is equivalent to the following question.
Question 1.

Given \( t \in [0, 1] \) and any unitarily invariant norm \( \| \cdot \| \), is it true that

\[
\| b_{\frac{1}{2}} \| \leq \| b_t \| \leq \| b_1 \|.
\]

The second inequality in Question 1 is the question raised by Bourin (2009) and it has been proved by Bhatia (2014) for the Hilbert–Schmidt norm under the condition \( \frac{1}{4} \leq t \leq \frac{3}{4} \).

It has been mentioned earlier that an affirmative answer to Bourin’s question in the trace norm has been given by Hayajneh et al. (2017).
Question 2.

Given \( t \in [0, 1] \) and any unitarily invariant norm \( \| \cdot \| \), is it true that

\[
\left\| k_1^{\frac{1}{2}} \right\| \leq \| k_t \| \leq \| k_1 \|?
\]

The first inequality in Question 2 can be stated as

\[
\left\| (A_1^{\frac{1}{2}} + B_1^{\frac{1}{2}})^2 \right\| \leq \| (A^t + B^t) (A^{1-t} + B^{1-t}) \|,
\]

which is a special case of the following more general form for commuting positive semidefinite matrices:

\[
\left\| (A_1^{\frac{1}{2}} B_1^{\frac{1}{2}} + A_2^{\frac{1}{2}} B_2^{\frac{1}{2}})^2 \right\| \leq \| (A_1 + A_2) (B_1 + B_2) \|,
\]

where \( A_1, A_2, B_1, B_2 \) are positive semidefinite matrices such that \( A_1 B_1 = B_1 A_1 \) and \( A_2 B_2 = B_2 A_2 \).
The inequality

\[ \left\| \left( A_1^{\frac{1}{2}} B_1^{\frac{1}{2}} + A_2^{\frac{1}{2}} B_2^{\frac{1}{2}} \right)^2 \right\| \leq \left\| \left( A_1 + A_2 \right) \left( B_1 + B_2 \right) \right\| \]

has been recently stated and proved by Audenaert (2015) for all unitarily invariant norms.

Hayajneh et al. (to appear) proved the following singular value inequality and majorization relations:
\[ s_j \left( (A^t + B^t) (A^{1-t} + B^{1-t}) \right) \leq 2^{3/2} s_j \left( (A^2 + B^2)^{1/2} \right) \text{ for } j = 1, 2, ..., n, \]

\[ s \left( (A^t + B^t) (A^{1-t} + B^{1-t}) \right) \prec_w 2s (A + B), \]

and

\[ s \left( A^t B^{1-t} + B^t A^{1-t} \right) \prec_w 2^{1/2} s \left( (A^2 + B^2)^{1/2} \right), \]

where \( A \) and \( B \) be positive semidefinite matrices and \( 0 \leq t \leq 1. \) Note that the previous singular value inequality and majorization relations are sharp.
As a consequence of the majorization relation

\[ s \left( (A^t + B^t)(A^{1-t} + B^{1-t}) \right) \prec_w 2s(A + B) \]

we have the inequality

\[ \|k_t\| \leq \|k_1\|. \]

This gives an affirmative answer to Question 2:

Given \( t \in [0, 1] \) and any unitarily invariant norm \( \|\| \cdot \|\| \) , is it true that

\[ \|k_\frac{1}{2}\| \leq \|k_t\| \leq \|k_1\| ? \]
The inequalities

\[(a^{\frac{1}{2}} + b^{\frac{1}{2}})^2 \leq (a^t + b^t) (a^{1-t} + b^{1-t}) \leq 2 (a + b)\]

have other matrix versions, which enable us to see the arithmetic-geometric mean inequality in a different new way in terms of $m_t$. 
Question 3.

Given $t \in [0, 1]$ and any unitarily invariant norm $\| \cdot \|$, is it true that

$$\| m_{1/2} \| \leq \| m_t \| \leq \| m_1 \|?$$

In fact, the second inequality in Question 3 has been proved by Plevnik (2016), while the first inequality can be stated as

$$\left\| \left( A_{1/2}^1 + B_{1/2}^1 \right)^2 \right\| \leq \left\| \left( A^t + B^t \right)^{1/2} \left( A^{1-t} + B^{1-t} \right) \left( A^t + B^t \right)^{1/2} \right\|,$$

which is a special case of the following more general form for commuting positive semidefinite matrices:

$$\left\| \left( A_{1/2}^1 B_{1/2}^1 + A_{1/2}^2 B_{1/2}^2 \right)^2 \right\| \leq \left\| \left( A_1 + A_2 \right)^{1/2} \left( B_1 + B_2 \right) \left( A_1 + A_2 \right)^{1/2} \right\|,$$

where $A_1, A_2, B_1, B_2$ are positive semidefinite matrices such that $A_1 B_1 = B_1 A_1$ and $A_2 B_2 = B_2 A_2$. 
The inequality

\[ \left\| \left( A_1^{1/2} B_1^{1/2} + A_2^{1/2} B_2^{1/2} \right)^2 \right\| \leq \left\| (A_1 + A_2)^{1/2} (B_1 + B_2) (A_1 + A_2)^{1/2} \right\| \]

has been recently stated and proved by Hayajneh et al. (2017) for all unitarily invariant norms.

This gives an affirmative answer to Question 3: Given \( t \in [0, 1] \) and any unitarily invariant norm \( \| \cdot \| \), is it true that

\[ \left\| m_1^{1/2} \right\| \leq \left\| m_t \right\| \leq \left\| m_1 \right\| ? \]
It should be mentioned here that the following question remains open:

**Open Question**

Given \( t \in [0, 1] \) and any unitarily invariant norm \( ||| \cdot ||| \), is it true that

\[
||| b_{\frac{1}{2}} ||| \leq ||| b_t ||| \leq ||| b_1 ||| ?
\]
In a recent paper, and in their investigations of the Lieb–Thirring trace inequalities, and in their attempt to answer Bourin’s questions, Hayajneh and Kittaneh (2013) proposed the following conjecture for commuting positive semidefinite matrices.

**Conjecture 1.**

Let $A_1, A_2, B_1, B_2$ be positive semidefinite matrices such that $A_1 B_1 = B_1 A_1$ and $A_2 B_2 = B_2 A_2$. Then for every unitarily invariant norm,

$$
\| |A_1 B_2 + A_2 B_1| | \leq \| |A_1 B_2 + B_1 A_2| | .
$$

An important special case of the previous inequality is the inequality

$$
\| |A^s B^p + B^q A^t| | \leq \| |A^s B^p + A^t B^q| | ,
$$

where $A, B$ are positive semidefinite matrices and $s, t, p, q$ are positive real numbers.
The Hilbert–Schmidt norm version of the previous inequality is the inequality
\[ \left\| A^s B^p + B^q A^t \right\|_2 \leq \left\| A^s B^p + A^t B^q \right\|_2. \]

Recently, Hayajneh et al. (2017) proved this inequality under the condition that
\[ \left| \frac{s}{s + t} - \frac{1}{2} \right| + \left| \frac{p}{p + q} - \frac{1}{2} \right| \leq \frac{1}{2}. \]

Replacing \( A \) and \( B \) by \( A^{\frac{1}{s+t}} \) and \( B^{\frac{1}{p+q}} \), we see that this inequality is equivalent to saying
\[ \left\| A^\mu B^v + B^{1-v} A^{1-\mu} \right\|_2 \leq \left\| A^\mu B^v + A^{1-\mu} B^{1-v} \right\|_2 \]
under the condition
\[ \left| \mu - \frac{1}{2} \right| + \left| v - \frac{1}{2} \right| \leq \frac{1}{2}. \]
Consequently, one can infer that
\[
\|b_t\|_2 \leq \|h_t\|_2 \quad \text{for} \quad \frac{1}{4} \leq t \leq \frac{3}{4}.
\]

Hayajneh et al. (2017) also generalized the inequality
\[
\|A^\mu B^\nu + B^{1-\nu}A^{1-\mu}\|_2 \leq \|A^\mu B^\nu + A^{1-\mu}B^{1-\nu}\|_2
\]
to complex values. In fact, they proved that the inequality
\[
\|A^w B^z + B^{1-\bar{z}}A^{1-\bar{w}}\|_2 \leq \|A^w B^z + A^{1-\bar{w}}B^{1-\bar{z}}\|_2
\]
holds for the complex numbers \(w, z\) under the condition
\[
\left| \text{Re } w - \frac{1}{2} \right| + \left| \text{Re } z - \frac{1}{2} \right| \leq \frac{1}{2}.
\]
Hayajneh et al. (to appear) also proved the following reverse-type inequality of the previous inequality:

\[ \| A^w B^z - B^{1-\bar{z}} A^{1-\bar{w}} \|_2 \geq \| A^w B^z - A^{1-\bar{w}} B^{1-\bar{z}} \|_2 \]

under the condition

\[ \left| \text{Re } w - \frac{1}{2} \right| + \left| \text{Re } z - \frac{1}{2} \right| \leq \frac{1}{2}. \]
Part 3: The Ando-Hiai-Okubo trace inequality

In their investigation of trace inequalities for multiple products of powers of two positive semidefinite matrices, Ando et al. (2000) proved that

\[ \text{tr} \left( A^{\frac{1}{2}} B \right)^2 \leq \text{tr} \ A^t BA^{1-t} B \leq \text{tr} \ AB^2 \]

for \( A, B \succeq 0 \) and \( 0 \leq t \leq 1 \).

Hayajneh et al. (to appear) generalized the previous inequalities by proving that the inequality

\[ \text{tr} \ A^s BA^{1-s} B \leq \text{tr} \ A^t BA^{1-t} B \]

holds for \( \frac{1}{2} \leq s \leq t \leq 1 \), where \( A \) is a positive semidefinite matrix and \( B \) is a Hermitian matrix.
It can be shown that the function $f(t) = \text{tr} \ A^t B A^{1-t} B$ is logarithmically convex (and hence it is convex) for $0 \leq t \leq 1$. Note that $f(t) = f(1 - t)$, and so $f(t)$ is symmetric about $t = \frac{1}{2}$. Thus, $f(t)$ is decreasing for $0 \leq t \leq \frac{1}{2}$, increasing for $\frac{1}{2} \leq t \leq 1$, attains its minimum at $t = \frac{1}{2}$, and attains its maximum at $t = 0$ and $t = 1$. 
Note that the previous trace inequality is equivalent to saying that

$$\text{tr} \ A^\alpha BA^\beta B \leq \text{tr} \ A^\gamma BA^\delta B$$

for $\alpha, \beta, \gamma, \delta \geq 0$ with $\alpha + \beta = \gamma + \delta$ and $\max \{\alpha, \beta\} \leq \max \{\gamma, \delta\}$. Here, $A$ is a positive semidefinite matrix and $B$ is a Hermitian matrix.

The second inequality in

$$\text{tr} \left( A^{\frac{1}{2}} B \right)^2 \leq \text{tr} \ A^t B A^{1-t} B \leq \text{tr} \ AB^2$$

is a particular case of the inequality

$$\left| \text{tr} \ A^s B^t A^{1-s} B^{1-t} \right| \leq \text{tr} \ AB,$$

where $A$ and $B$ are positive semidefinite matrices and $0 \leq s, t \leq 1$. 
Ando et al. (2000) proved that the previous inequality holds for all non-negative real numbers $s, t$ for which

$$\left| s - \frac{1}{2} \right| + \left| t - \frac{1}{2} \right| \leq \frac{1}{2}.$$ 

It is natural to ask what is the complete range of validity of the inequality

$$\left| \text{tr} \ A^s B^t A^{1-s} B^{1-t} \right| \leq \text{tr} \ AB?$$

Plevnik (2016) gave a counterexample to the previous inequality. He answered the question in the negative for $s = \frac{4}{5}, t = \frac{1}{5}$. 
Recently, Hayajneh et al. (2017) generalized the previous inequality by proving that the inequality

$$|\text{tr} \ A^w B^{1-w} A^{1-w} | \leq \text{tr} \ AB$$

holds for all complex numbers $w, z$ for which

$$\left| \text{Re} \ w - \frac{1}{2} \right| + \left| \text{Re} \ z - \frac{1}{2} \right| \leq \frac{1}{2}.$$

A special case of the previous inequality when $w = z$ is the inequality

$$|\text{tr} \ A^z B^z A^{1-z} B^{1-z} | \leq \text{tr} \ AB.$$

Bottazzi et al (2015) proved this inequality under the condition that

$$\frac{1}{4} \leq \text{Re} \ z \leq \frac{3}{4}.$$
As an application of the inequality

\[ \text{tr } A^s B A^{1-s} B \leq \text{tr } A^t B A^{1-t} B \text{ for } \frac{1}{2} \leq s \leq t \leq 1, \]

Hayajneh et al. (to appear) proved the following Hilbert-Schmidt norm inequality

\[ \| A^s B + B A^{1-s} \|_2 \leq \| A^t B + B A^{1-t} \|_2 \]

for \( \frac{1}{2} \leq s \leq t \leq 1 \), where \( A \) is a positive semidefinite matrix and \( B \) is a Hermitian matrix.
As a second application of the inequality

\[ \text{tr } A^s BA^{1-s} B \leq \text{tr } A^t BA^{1-t} B \text{ for } \frac{1}{2} \leq s \leq t \leq 1, \]

we have the following trace inequality

\[ \text{tr } A^t BA^{1-t} (\log A) B \leq \text{tr } A^t (\log A) BA^{1-t} B \]

for \( \frac{1}{2} \leq t \leq 1 \), where \( A \) is a positive definite matrix and \( B \) is a Hermitian matrix.
As a consequence of the previous trace inequality, we obtain the inequality

$$\|A^t B + BA^{1-t} \log A\|_2 \leq \|A^t (\log A) B + BA^{1-t}\|_2$$

for $\frac{1}{2} \leq t \leq 1$, where $A$ is a positive definite matrix with $\sigma(A) \subseteq [e^{-1}, 1] \cup [e, \infty)$ and $B$ is a Hermitian matrix.
It would be interesting to investigate the following conjectures concerning the generalizations of the previous Hilbert-Schmidt norm inequalities to the wider class of unitarily invariant norms.

**Conjecture 2.**

Let $A$ be a positive semidefinite matrix and $B$ be a Hermitian matrix. Then for $\frac{1}{2} \leq s \leq t \leq 1$ and for every unitarily invariant norm, we have

$$\| A^s B + B A^{1-s} \| \leq \| A^t B + B A^{1-t} \| .$$

This conjecture has been affirmatively settled for the case $s = \frac{1}{2}$ and $t = 1$. That is,

$$\| A^{\frac{1}{2}} B + B A^{\frac{1}{2}} \| \leq \| AB + B \| .$$
Conjecture 3.

Let $A$ be a positive definite matrix such that $\sigma(A) \subseteq [e^{-1}, 1] \cup [e, \infty)$ and $B$ be a Hermitian matrix. Then for $\frac{1}{2} \leq s \leq t \leq 1$ and for every unitarily invariant norm, we have

$$||| A^t B + BA^{1-t} \log A ||| \leq ||| A^t (\log A) B + BA^{1-t} |||.$$


• L. Plevnik, On a matrix trace inequality due to Ando, Hiai and Okubo, Indian J. Pure Appl. math. 47 (2016), 491-500.
Thank you for your attention!